Supplementary Material for Generalized Belief Propagation on Tree Robust Structured Region Graphs

1 Supplementary Material

The following properties of Loop-SRGs are proven in [1]:

THEOREM **1.** *A Loop-SRG has $\sum_R \kappa_R = |L| - |E| + |V|$, where $|L|$ is the number of loop regions, $|E|$ the number of edge regions and $|V|$ the number of node regions.*

THEOREM **2.** *A Loop-SRG is singular if $\sum_R \kappa_R > 1$.*

THEOREM **3.** *A Loop-SRG is singular iff there is a subset of loop regions and constituent edge regions such that all of the edge regions have 2 or more parents.*

The following proofs make use of these theorems as well as the reduction operators presented in [1].

Theorem 4: A Loop-SRG is Non-Singular and satisfies Counting Number Unity if its loop outer regions are a Fundamental Cycle Basis (FCB) of G.

Proof. ($FCB \Rightarrow \sum_R \kappa_R = 1$) From Theorem 2, we see that $\mu = E - V + 1$ is exactly the number of loops needed to ensure that $\sum_R \kappa_R = 1$. From this it follows that a loop SRG will satisfy counting number unity if the set of loop outer regions form a cycle basis of G.

($FCB \Rightarrow$ non-singularity) First remove all factors from the outer regions. Let $\mathcal{B}$ be a FCB of G and map each outer region R to one of the cycles in this basis. Since $\mathcal{B}$ is fundamental, there exists some ordering π such that cycle $C_{\pi(i)}$ has some edge that does not appear in any cycle preceding it. Let $R_{\pi(i)}$ be the loop outer region corresponding to cycle $C_{\pi(i)}$ and let $E_{\pi(i)}$ be the edge(s) unique to cycle $C_{\pi(i)}$. Let $R_{E_{\pi(i)}}$ be the edge region corresponding to edge $E_{\pi(i)}$. Since $E_{\pi(i)}$ is unique to $C_{\pi(i)}$, edge region $R_{E_{\pi(i)}}$'s only parent is $R_{\pi(i)}$. Thus, edge region $R_{E_{\pi(i)}}$ can be *Dropped*.

Let $\mathcal{C}(R_{\pi(i)})$ be the set of cliques of outer region $R_{\pi(i)}$. The clique corresponding to edge $E_{\pi(i)}$ can be *Shrunk* since child region $R_{E_{\pi(i)}}$ was dropped. Let $\bar{E}_{\pi(i)} = C_{\pi(i)} \setminus E_{\pi(i)}$ be the set of edges not unique to $C_{\pi(i)}$. The *Shrink* operation leaves the structure $\mathcal{G}(R_{\pi(i)})$ of region $R_{\pi(i)}$ as a chain over the edges $\bar{E}_{\pi(i)}$. This chain can be *Split* into its constituent edges by choosing the variables not in edge $E_{\pi(i)}$ as separators. The *Split* operation produces a set of edge outer regions $\mathcal{R}_{\bar{E}_{\pi(i)}}$ and node regions $\mathcal{R}_{\bar{V}_{\pi(i)}}$. These edge and node regions are duplicates of regions already in the SRG. And since all factors were initially removed, the regions in $\mathcal{R}_{\bar{E}_{\pi(i)}}$ and $\mathcal{R}_{\bar{V}_{\pi(i)}}$ can then be merged with the regions that they duplicate.

The loop outer regions can be *reduced* in this way along the ordering π - i.e. beginning with cycle $C_{\pi(\mu)}$ and ending with cycle $C_{\pi(i)}$. Reducing all loop regions yields an acyclic SRG (comprised of edge and node regions) which is non-singular from Theorem 5 in [1].

□

Theorem 5: A Loop-SRG is Tree Robust if its loop outer regions are a Tree Robust cycle basis of G.

Proof. In proving Theorem 4 the idea was to show that a loop outer region can be *reduced* if it contains a *unique* edge; the fact that the loops form a FCB means that the loops can be reduced in an order such that each loop has a unique edge. However, reducing a loop region to its constituent edges requires *Merge* operations that can only be performed if the outer regions have no factors. This condition was guaranteed in the proof of Theorem 4 by initially removing all factors from the SRG. In a Tree Robust SRG, only a subset of the factors are removed. Thus, we need a stronger tool. Using the *Factor Move* operator it is easy to show that:

LEMMA **1.** *A loop outer region can be reduced if it contains at least one unique edge not covered by a factor.*

The desired result follows by incorporating this Lemma into the same sequence of reduction operators used in the proof of Theorem 4. □

To simplify notation, in the following proofs let $G_{\pi(i)} = \{C_{\pi(1)}, ..., C_{\pi(i)}\}$.

LEMMA **2.** *A TR basis $\mathcal{B}$ is fundamental.*

Proof. (Proof by contrapositive): Assume that $\mathcal{B}$ is not fundamental. Then there exists no ordering π of the cycles in $\mathcal{B}$ such that $C_{\pi(i)} \setminus G_{\pi(i-1)} \neq \emptyset$ for $2 \leq i \leq \mu$. This implies that there is no ordering π for which $\left\{C_{\pi(i)} \setminus G_{\pi(i-1)}\right\} \setminus T \neq \emptyset$ for any spanning tree T. Thus, the basis is not TR. $\square$

Theorem 6: Let $\mathfrak{B}^{|k|}$ denote all size k subsets of cycles in $\mathcal{B}$. A FCB $\mathcal{B}$ is Tree Robust iff $I(\mathcal{B}_k)$ is cyclic and not-empty for all $\mathcal{B}_k \in \mathfrak{B}^{|k|}$ for $1 \leq k \leq \mu$.

Proof. ($I(\mathcal{B}_k)$ is cyclic for all subsets of $\mathcal{B} \Rightarrow$ TR) First, we note that since the unique edge graph is cyclic for all subsets of cycles, the unique edge graph is cyclic for all partial orderings of the cycles as well.

Let $\mathcal{B}_{\pi(i)} = \mathcal{B} \setminus \{C_{\pi(i+1)}, ..., C_{\pi(\mu)}\}$ denote the set of cycles not appearing in the partial order $\pi(i+1), ..., \pi(\mu)$.

A basis is not TR if $\exists$ some j ($2 \leq j \leq \mu$) such that $\{C \setminus G_{\pi(j-1)}\} \setminus T = \emptyset$ for all $C \in \mathcal{B}_{\pi(j-1)}$ for all orders $\pi \in \Pi$. We show that this cannot occur given that $I(\mathcal{B}_{\pi(j)})$ is cyclic for all $\pi \in \Pi$.

For $\{C \setminus G_{\pi(j-1)}\} \setminus T = \emptyset$ for all $C \in \mathcal{B}_{\pi(j-1)}$ and all orders, we require that either: 1) $C \setminus G_{\pi(j-1)} = \emptyset$; or 2) $C \setminus G_{\pi(j-1)}$ be acyclic. Since $I(\mathcal{B}_{\pi(j)})$ is not empty for all orderings, there must exist some $C \in \mathcal{B}_{\pi(j)}$ such that $C \setminus G_{\pi(j-1)} \neq \emptyset$. And since $I(\mathcal{B}_{\pi(j)})$ is cyclic for all orderings, there cannot exist some tree T that covers all edges in $I(\mathcal{B}_{\pi(j)})$.

(TR $\Rightarrow I(\mathcal{B}_k)$ is cyclic for all subsets of $\mathcal{B}$). Assume that $I(\mathcal{B}_k)$ is acyclic and consider some spanning tree T that 'covers' all of the edges in $I(\mathcal{B}_k)$ (i.e. $I(\mathcal{B}_k) \setminus T = \emptyset$). Clearly the basis $\mathcal{B}$ would not be tree exact w.r.t. to T and therefore not TR. We now must show that there exists some ordering such that $\mathcal{B}_{\pi(k)} = \mathcal{B}_k$. Assume that such an ordering does not exist. Then there must exist some $j > k$ for which $C \setminus G_{\pi(j)} = \emptyset$ for all $C \in \mathcal{B}_{\pi(j)}$. This would mean that the basis is not fundamental. However, from the previous Lemma we know that if $\mathcal{B}$ is not fundamental, it is not TR. $\square$

Corollary 1: An FCB is TR iff $\mathcal{B}_k$ is TR for all $\mathcal{B}_k \in \mathfrak{B}^{|k|}$ for $1 \leq k \leq \mu$.

Proof. ($\mathcal{B}$ is TR $\Rightarrow \mathcal{B}_k$ is TR for all k) Assume there is some $\mathcal{B}_k \subseteq \mathcal{B}$ that is not TR. Then there exists some $I(\mathcal{B}_k)$ that acyclic. We know that $\mathcal{B}$ is TR iff $I(\mathcal{B}_k)$ is cyclic for all subsets of $\mathcal{B}$. Therefore, $\mathcal{B}$ is not TR. ($\mathcal{B}_k$ is TR for all k $\Rightarrow \mathcal{B}$ is TR) Follows immediately from proof of Theorem 6. $\square$

References

[1] M. Welling, Tom Minka, and Yee Whye Teh. Structured region graphs: Morphing EP into GBP. In *Proc. of the Conf. on Uncertainty in Artificial Intelligence*, pages 607–614, 2005.