Stopping Rules for Randomized Greedy Triangulation Schemes

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Abstract

Many algorithms for performing inference in graphical models have complexity that is exponential in the treewidth - a parameter of the underlying graph structure. Computing the (minimal) treewidth is NP-complete, so stochastic algorithms are sometimes used to find low width tree decompositions. A common approach for finding good decompositions is iteratively executing a greedy triangulation algorithm (e.g. min-fill) with randomized tie-breaking. However, utilizing a stochastic algorithm as part of the inference task introduces a new problem - namely, deciding how long the stochastic algorithm should be allowed to execute before performing inference on the best tree decomposition found so far. We refer to this dilemma as the *Stopping Problem* and formalize it in terms of the total time needed to answer a probabilistic query. We propose a rule for discontinuing the search for improved decompositions and demonstrate the benefit (in terms of time saved) of applying this rule to Bayes and Markov network instances.

Introduction

Many algorithms for performing inference in graphical models require finding a tree decomposition of the model's underlying graph structure (Dechter 1999; Lauritzen and Spiegelhalter 1988). Such methods have complexity that is exponential in the width of the tree decomposition - a quantity known as the treewidth. Finding a low width decomposition is thus a critical component of any inference task.

Unfortunately, finding a minimum width tree decomposition is NP-complete (Arnborg, Corneil, and Proskourowski 1987). For this reason, stochastic local search algorithms are sometimes used to find low width tree decompositions (Clautiaux et al. 2004; Kjærulff 1992; Larranaga et al. 1997). Standard greedy triangulation algorithms, such as min-fill, can also be viewed as stochastic search algorithms when tie-breaking is randomized. In fact, iteratively executing the min-fill algorithm with random tie-breaking has been shown to work well (Fishelson and Geiger 2003). However, applying a stochastic algorithm as part of an inference procedure introduces a new dilemma: how long should the stochastic algorithm be allowed to execute before performing inference on the best decomposition found so far?

Since query complexity is exponentially related to the induced width, investing the time to search for improved tree decompositions may lead to a drastic reduction in the run-time of an inference procedure. We refer to the dilemma of how long to seek good decompositions for as the *Stopping Problem* because of its similarity to the problem of deciding when to stop software testing (Ross 1985). The stopping problem is important when minimizing the time to respond to a query. It is also important when a query must be answered within a certain amount of time, as the portion of time spent seeking good decompositions can affect response accuracy as well.

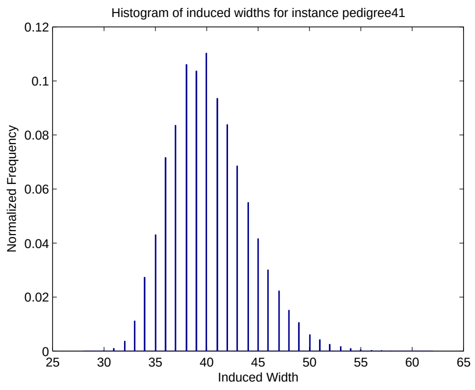
In this paper, we formalize the stopping problem in terms of the total time needed to answer a probabilistic query. Doing so requires characterizing the run-time behavior of an inference procedure that includes a stochastic component - a difficult task because of the algorithm's inherent randomness. Hoos and Stutzle analyzed the behavior of WalkSAT, a stochastic local search algorithm for propositional satisfiability (SAT) problems, and empirically found it's run-time to be exponentially distributed(1999). Their analysis characterized the time WalkSAT requires to find a solution on a random set of 3-SAT instances.

In this paper, we present an approach for estimating the total run-time of an inference procedure that is instance-specific. We then propose a stopping rule that specifies when to terminate execution of the stochastic component of the inference procedure. Fishelson and Geiger suggest terminating the stochastic component when the rate of occurrence of improved tree decompositions falls below a threshold (2003). In practice, several ad hoc rules are used to discontinue the search for improved tree decompositions. Notably, the winning solver in two of the three tasks at the UAI 2010 approximate inference competition ran a stochastic variant min-fill a fixed number of times on every problem instance[1].

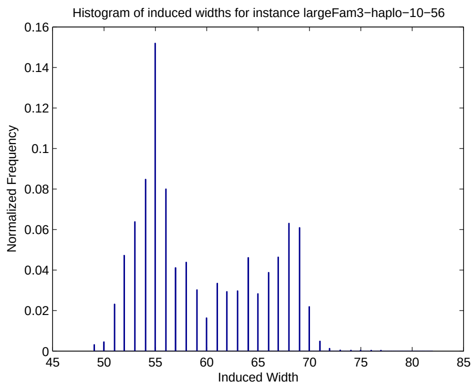
To motivate the importance of the stopping problem, consider the histograms in Figure 1. These histograms were generated by running min-fill with random tie-breaking $100,000$ times and recording the induced width of the tree decomposition found in each iteration. These histograms

[1] `http://www.cs.huji.ac.il/project/UAI10/`.

represent the typical variability in induced width observed from running min-fill on linkage analysis instances[2]. The two empirical distributions in Figure 1 have very different shapes. The distribution of instance pedigree41 is more heavily tailed than that of instance largeFam3-haplo-10-56. This suggests that stochastic search will be more likely to find a low width tree decomposition on largeFam3-haplo-10-56 than on pedigree41. As such, more time should be invested seeking low width decompositions on instance pedigree41. The stopping problem involves deciding how much time to invest on a new instance, with unknown distribution.



(a) Pedigree41 (1062 variables))



(b) largeFam3-haplo-10-56 (2297 variables)

Figure 1: Histograms of induced widths from 100K iterations of min-fill with random tie-breaking.

The main contribution of this paper is to formulate the *Stopping Problem* and formalize the trade-off between searching for improved tree decompositions using a stochastic greedy triangulation algorithm and exploiting the best decomposition found so far. The Stopping Problem is presented in Section 3, along with a novel Stopping Rule - the Minimum Total Query Time (MTQT) rule. The fourth section contains an empirical evaluation of the MTQT stopping rule and typical ad hoc stopping rules. The evaluation demonstrates the benefit, in terms of time saved, of applying the MTQT rule.

[2]Empirical distributions of more problems can be found at `http://www.ics.uci.edu/~agelfand/elim.html`.

Background

This section contains some necessary definitions.

DEFINITION **1.** ***Graphical Model*** *- A Graphical model $\mathcal{R}$ is a 4-tuple $\mathcal{R} = \langle X, D, F, \otimes \rangle$, where:*

1. $\mathbf{X} = \{X_1, ..., X_n\}$ *is a set of variables;*
2. $\mathbf{D} = \{D_1, ..., D_n\}$ *is the respective set of finite domains;*
3. $\mathbf{F} = \{f_1, ..., f_r\}$ *is a set of real-valued functions defined over a subset of variables $S_i \subseteq X$. The scope of function f_i, denoted scope(f_i), is its set of arguments, S_i;*
4. $\otimes_i f_i \in \{\prod_i f_i, \sum_i f_i, \bowtie_i f_i\}$ *is a combination operator.*

A graphical model is the combination of functions: $\otimes_{i=1}^{r} f_i$.

Let $G(\mathbf{X}, \mathbf{E})$ denote an undirected graph, where $\mathbf{X}$ is the set of nodes in the graph and $\mathbf{E}$ is the set of edges in the graph. The *primal graph G* of a graphical model associates a node with each variable and connects any two nodes whose variables appear in the same scope. All the graphs considered in this paper are undirected and simple. Let $n_G(X)$ denote the set of nodes that are neighbors of X in G and $\tilde{n}_G(X)$ the neighbors of X including X. A *clique* of G is a set of nodes in G such that all the nodes are neighbors.

Eliminating a node X from G is a process resulting in the creation of another graph G' by: 1) adding (fill-in) edges to make $n_G(X)$ a clique; and 2) removing X and its incident edges from G. The number of fill-in edges needed to make $n_G(X)$ a clique is the *fill-in elimination cost* of node X.

DEFINITION **2.** ***Elimination Order*** *- An* elimination order *π is a total ordering of the nodes of G. π induces a sequence of graphs $G_1 = G, G_2, ...G_n$, where G_{i+1} is obtained from graph G_i by* eliminating *node $\pi(i)$, the i^{th} node in the elimination order. π also induces a sequence of cliques $\mathbf{C}_1, ..., \mathbf{C}_n$ where $\mathbf{C}_i$ is the clique consisting of $\tilde{n}_G(\pi(i))$.*

A tree decomposition can be constructed from an elimination order by connecting the cliques $\mathbf{C}_1, ..., \mathbf{C}_n$ so as to satisfy the running intersection property (see e.g. (Bodlaender and Koster 2010) for details).

DEFINITION **3.** ***Induced Width*** *- The induced width of elimination order π wrt graph G is*

$$w(\pi, G) \equiv \max_{i=1...n} |\mathbf{C}_i| - 1 \tag{1}$$

The treewidth *is the minimum induced width ordering across all permutations of the nodes in G.*

DEFINITION **4.** ***Total State Space*** *- The* total state space *of an elimination order π wrt a graphical model $\mathcal{R}$ is:*

$$s(\pi, \mathcal{R}) = \sum_{i=1}^{n} s(\pi(i), G_i) \tag{2}$$

where $s(\pi(i), G_i) = \prod_{X \in \tilde{n}_{G_i}(\pi(i))} |D_X|$ is the space required to eliminate node i from graph G_i and D_X is the domain of node X (Kjærulff 1992).

The *min-fill heuristic* is a greedy algorithm for constructing elimination orders. It operates as follows. Select the node x whose elimination requires adding the fewest number of edges (i.e. is of minimum fill-in elimination cost). Place this node in $\pi(1)$ and eliminate it from the graph. Select a remaining node with minimum fill-in elimination cost, place it in the next position in the order and eliminate it. Repeat until all nodes have been eliminated. More than one node may have the minimum fill-in elimination cost in each step. The stochastic min-fill algorithm breaks such ties by randomly selecting from the set of minimum cost nodes.

The fill-in elimination cost is just one criteria for selecting the node eliminated in each step. Schemes that greedily build elimination orders using other criterion were studied in (Kjærulff 1992; Bodlaender and Koster 2010) and more recently in (Kask et al. 2011).

The Stopping Problem

In this section we formulate the stopping problem. Let $T_c(\pi, \mathcal{R})$ denote the time needed to compute a quantity (e.g. a posterior marginal) in a Bayes or Markov network instance $\mathcal{R}$ using elimination order π. We assume that the computation time is a deterministic quantity that depends on the inference algorithm used. For example, in the experiments section T_c is taken with respect to the Bucket Elimination (BE) algorithm, whose performance is typical of most exact inference algorithms (Dechter 1999).

Ultimately our goal is to minimize T_c. Since searching over all possible orderings is infeasible, we utilize a stochastic ordering heuristic (SOH), such as min-fill with random tie-breaking, to search for small computation cost elimination orders.

Executing a SOH for a period of time may yield improved elimination orders. However, the time needed to find an improved ordering is uncertain. Let $T_s(\pi, \mathcal{R})$ be a random variable describing the amount of time the SOH will take to find ordering π. The total time to answer a probabilistic query, T_q is the sum of the random variable $T_s(\pi, \mathcal{R})$ and the deterministic quantity $T_c(\pi, \mathcal{R})$. T_q is thus a random variable expressed as:

$$T_q(\pi, \mathcal{R}) = T_s(\pi, \mathcal{R}) + T_c(\pi, \mathcal{R}) \tag{3}$$

DEFINITION **5.** ***Stopping Problem*** *- Assume that we execute a SOH iteratively. Let t_i denote the time that iteration i of the SOH completes and let π_i be the elimination order found in the i^{th} iteration. Let $T_q^{(i)} = t_i + T_c^{(i)}$ denote the minimum total query time through iteration i, where $T_c^{(i)} = \min_{j=1}^{i} T_c(\pi_j, \mathcal{R})$ is the minimum computation time through iteration i. The stopping problem is to identify the index i_{stop} such that*

$$T_q^{(i_{stop})} = \min_i T_q^{(i)} = \min_\pi T_q(\pi, \mathcal{R}) \tag{4}$$

We now present three rules for identifying i_{stop}.

Min Total Query Time (MTQT) Stopping Rule

The Minimum Total Query Time (MTQT) stopping rule operates by making an estimate of the expectation $E[T_s(\pi, \mathcal{R})]$. Combining this estimate with $T_c(\pi, \mathcal{R})$ leads to a stopping rule based on the expected total query time.

Since many inference algorithms have complexity that is captured well by the induced width of an elimination order, it is plausible to express the total query time as

$$T_q(w) = T_s(w) + T_c(w) \tag{5}$$

where $w \equiv w(\pi, G)$ is the induced width of ordering π in the primal graph G of instance $\mathcal{R}$. By making this simplification, we assume that all orderings of induced width w have the same computation time. The ramifications of this assumption are discussed in a following section.

Let $t_i \geq 0$ be the time that iteration i of the SOH completes. We model each completion of the SOH as an event in a Poisson process whose rate is λ.[3] Let w_i be the induced width of elimination order π_i. Let W be the set of feasible[4] widths and let $W(j)$ denote the j^{th} element in W ($|W| = m$). Let π_i be of width $W(j)$ with probability p_j ($\sum_{j=1}^m p_j = 1$). If w_i is independent of the previous orderings found, then each width $W(j)$ will occur as an independent Poisson process with rate $\lambda_j = \lambda p_j$ (Ross 2006). The mean arrival time of an ordering of width $W(j)$ is:

$$E\left[T_s(W(j))\right] = 1/\lambda_j = 1/(p_j \cdot \lambda) \tag{6}$$

λ and $\mathbf{p} = p_1, ..., p_m$ are unknown and must be estimated.

The Maximum Likelihood estimate for the rate of a Poisson process when n events are observed by time t_n is:

$$\hat{\lambda}_n = n/t_n. \tag{7}$$

Let X_j^n be the number of times an ordering of width $W(j)$ was observed in n iterations of the SOH. Then the vector $(X_1^n, ..., X_m^n)$ follows a Multinomial distribution with parameters n,$\mathbf{p}$. The posterior mean of p_j at iteration n is:

$$\hat{p}_{j,n} = E\left[p_j | w^{(n)}\right] = \frac{X_j^n + \alpha_j}{n + \alpha_0} \tag{8}$$

where α_j are the parameters of a Dirichlet prior $Dir(\alpha_1, ..., \alpha_m)$ and $\alpha_0 = \sum_{j=1}^m \alpha_j$.

The predicted total query time for a width $W(j)$ ordering given i iterations of the SOH is:

$$\begin{aligned} E[T_q(W(j))|w^{(i)}, t_i] &= E[T_s(W(j))] + T_c(W(j)) && (9) \\ &= 1/(\hat{\lambda}_i \cdot \hat{p}_{j,i}) + T_c(W(j)) && (10) \end{aligned}$$

where $w^{(i)} = \{w_1, ..., w_i\}$ and T_c is assumed to be known.

The MTQT stopping rule is described in the algorithm on the following page.

Minimum Improvement (MI) Rule

The Minimum Improvement (MI) stopping rule is the procedure proposed in (Fishelson and Geiger 2003). This approach monitors the elimination orders found by the SOH and terminates when the occurrence of improved orderings becomes too infrequent.

[3]The hypothesis that iterations of the SOH complete occur as events of a Poisson process was empirically justified by fitting an exponential distribution to the SOH execution times in several problem instances.

[4]$W = \{lb, ..., ub\}$ is found by computing a lower bound lb using the minor-min-width heuristic and an upper bound ub by running the min-fill heuristic once prior to iteratively running the SOH.

```
The MTQT Stopping Rule
Input: Graphical Model $\mathcal{R}$, feasible widths $W$ and Dirichlet prior parameters $\alpha_1, ..., \alpha_m$
Output: stopping index $i_{stop}$
Initialization: $T_c^{(0)} = \infty$, done = false, $i = 1$
  while done == false do
    $(\pi_i, t_i) =$ SOH()
    if $T_c(\pi_i, \mathcal{R}) < T_c^{(i-1)}$ then
      $T_c^{(i)} = T_c(\pi_i, \mathcal{R})$
    end if
    Update $\hat{\lambda}$ with Eqn.7
    done = true
    for $j = 1$ to $m$ do
      Update $\hat{p}_j$ with Eqn.8
      if $T_c(W(j)) + (\hat{\lambda} \cdot \hat{p}_j)^{-1} < T_q^{(i)}$ then
        done = false
      end if
    end for
    $i = i + 1$
  end while
  $i_{stop} = i$
```

The MI procedure operates as follows. The min-fill algorithm is run once to establish a baseline computation time on the problem instance. The SOH is then run iteratively and the improvement with respect to the baseline ordering - the improvement ratio - is computed after each iteration. The SOH is terminated when this ratio falls below some threshold (β) or a maximum number of iterations has been reached. In terms of total query time, the MI rule stops the SOH at the first iteration i_{stop} in which:

$$\frac{T_q^{(i_{stop})}}{T_q(\pi_1, R)} = \frac{t_{i_{stop}} + T_c^{(i_{stop})}}{t_1 + T_c(\pi_1, R)} \leq \beta \qquad (11)$$

MinMult Rule

This procedure utilizes the following ad hoc rule for terminating the SOH: if it took i iterations to find the current best elimination order, allow an additional $a \times i$ iterations to find a better ordering. The procedure is governed by two parameters - a and b. a is a positive multiplication factor and $b > 0$ is an integer threshold value that specifies the minimum number of times the SOH will be run.

Approximating $T_c(w)$ and $T_c(s)$

In formulating the stopping problem, we assume that $T_c(\pi, \mathcal{R})$ is a known deterministic quantity. In the MTQT section, we proposed to express T_c as a function of w. Since orderings with the same induced width may have different computation times, this simplification prevents us from precisely computing T_c for a given ordering. As a result, we approximate $T_c(w)$ by a model of the following form:

$$T_c(w) = a_0 \cdot n_R \cdot k_R^{(a_1 + a_2 \cdot w)} \qquad (12)$$

where n_R is the number of variables in instance $\mathcal{R}$, k_R is the average domain size and the a_i's are model parameters. This model follows from the computational complexity of the BE algorithm (Dechter 1999).

The model was fit as follows. First, we recorded the actual run-time of BE on 1000 different elimination orders from a small set of Markov network instances. These run-times were treated as observations and $\hat{a}_i$'s were found by minimizing the sum of squared error.

$T_c(s)$, the computation time given the total state space of an ordering (see Definition 4) was approximated in a similar fashion using the model:

$$T_c(s) = a_0 + a_1 \cdot s \qquad (13)$$

This model does not contain instance specific parameters.

Experimental Results

This section contains an empirical evaluation of the proposed stopping rules. We describe the experimental setup, the evaluation methodology and then discuss results.

Experimental Setup

We ran two sets of experiments to evaluate the stopping rules. In the first set, query times were expressed using induced width - i.e. $T_q(w(\pi, \mathcal{R}))$. In the second set, they were expressed using total state space - i.e. $T_q(s(\pi, \mathcal{R}))$.

The experiments were run on the set of Bayes and Markov network instances shown in Table 1. Instances from the pedigree and mastermind problem classes were used in the solver competition held at the UAI-2008 conference. Instances in the largeFam3 problem class are derived from genetics. The third column of Table 1 contains the average number of variables in each problem class. The fourth column contains the average domain size, which may be less than 2 due to evidence. Standard deviations are shown in parentheses.

The parameters of each stopping rule were chosen via 10-fold cross-validation within each problem class. In the MTQT procedure, we assumed each of the W possible widths were equally likely - i.e. $\alpha_1 = \cdots = \alpha_m = \alpha$ in the Dirichlet prior. The value of β in the MI rule was set in the range $1e^{-4}, ..., 1e^{-8}$. The parameters of the MinMult rule were chosen by considering all pairs (a b), where $a \in A = \{1...5\}$ and $b \in B = \{100, 500, 1000, 5000, 10000\}$. The evaluation also includes a baseline stopping rule in which the min-fill algorithm is terminated after a fixed number of iterations. For the fixed rule, i_{stop} was chosen from: $100, 500, 1000, 5000, 10000, 50000$.

Evaluation Methodology

The stopping rules were evaluated as follows. The min-fill algorithm with random tie-breaking was run $N = 100,000$

Problem Class	# Inst.	$\bar{n}$	$\bar{\lvert D \rvert}$
pedigree	22	917.1 (315.5)	2.0 (0.15)
largeFam3	122	3513.4 (806.6)	1.54 (0.06)
mastermind	128	2158.8 (762.9)	2.0 (0.0)

Table 1: Summary of problem instances used in the experimental evaluation.

times on each problem instance to generate a sequence of induced widths $w_1, ..., w_N$ and state space sizes $s_1, ..., s_N$. The times that each min-fill iteration completed $t_1, ..., t_N$ were also recorded. Given this sequence of observations, we determined i_{stop} using each of the stopping rules. In practice, the stopping rules operate in an on-line fashion and do not have access to all N observations. However, for evaluative purposes we require the full sequence of orderings. We then identify the index $i^\star = \arg\min_{i=1}^{N} T_q(\pi_{(i)}, \mathcal{R})$ at which the total query time was minimized. $i^\star$ is found using the estimates of $T_c(w)$ ($T_c(s)$) described previously.

Figure 2 illustrates the difference between $i^\star$ and i_{stop}. The figure contains a scatter plot of the induced widths found in each iteration. The thick line traces the minimum width found through each iteration. The solid circle at $i_{stop} = 15296$ indicates the stopping index predicted by the MTQT rule. The solid square indicates the best possible stopping iteration at $i^\star = 11090$. Table 2 illustrates the total query time computation on some pedigree instances.

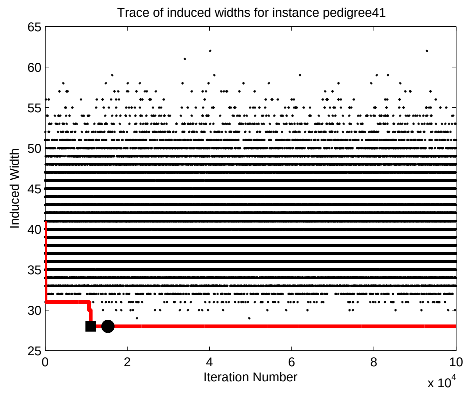


Figure 2: Trace of the induced width of the elimination orders found by executing min-fill 100K times on instance pedigree41. i_{stop} predicted by the MTQT rule is indicated by the solid circle. $i^\star$ is indicated by a solid square.

The stopping rules were evaluated using two different metrics. The first metric considered is the seek time error:

$$T_{Err} = t_{i_{stop}} - t_{i^\star} \tag{14}$$

which is the difference between the amount of time (in seconds) the SOH executed before being terminated by the stopping rule and the amount of time the SOH needed to execute to find the minimum width (total state space) ordering. The seek time error indicates how accurately the stopping rule predicts the occurrence of the minimum width (total state space) ordering. The sign of T_{Err} provides an indication as to whether a stopping rule is under(over) estimating the occurrence of the minimum width ordering. The second metric considered is the slow-down ratio:

$$SDR = T_q^{(i_{stop})} \,/\, T_q^{(i^\star)} \tag{15}$$

This metric measures how much slower the total query response time is using a stopping rule than the smallest possible response time. The SDR measures how the choice of stopping index impacts the total query response time.

Sample Total Query Time Computations

Instance	i_{stop}	$\log(T_q^{(i_{stop})})$	$i^\star$	$\log(T_q^{(i^\star)})$
pedigree18	70	7.32	102	6.54
pedigree34	26494	13.58	39490	12.09
pedigree7	27851	13.54	4239	13.53

Table 2: Total Query Time computations on selected pedigree instances (pedigree18 1184 variables; pedigree34 1160 vars; pedigree7 1068 vars).

Results

The results are organized into two different tables. Table 3 contains results from the experiments where T_q was expressed in terms of induced width and Table 4 contains the total state space results. Each table is organized by problem class. The first row in each class, contains the parameter settings of each stopping rule. The second and fourth rows contain the seek time error and slow-down ratio of each stopping rule, averaged across the instances in a class. Standard deviations are shown in parentheses.

The MTQT stopping rule performs quite well. In the total state space experiments the MTQT rule had a smaller average SDR on all three problem classes. On the mastermind instances, which have an average computation time of 640 hours[5], the difference between the SDR of the MTQT rule (1.83) and the SDR of the MinMult rule (1.97) amounts to an average savings of nearly 90 hours!

In the induced width experiments, the MTQT rule, fixed rule and MinMult rule all yield comparable average SDR on the pedigree and mastermind instances. The MTQT rule yields the smallest average slow-down ratio on the large-Fam3 instances, but does so by having a much larger, positive seek time error than the other methods. This suggests that there is a natural trade-off between minimizing T_{Err} and SDR. On instances with large induced width (total state space) T_{Err} accounts for just a small portion of the total query response time. Thus, if the goal is to minimize SDR, it is generally better to overestimate the time at which the minimum width (space) ordering will occur.

We close the experiments section by noting two ways in which the performance of the MTQT procedure could be improved. First, in our implementation of the MTQT rule we assume that all widths in W, the set of feasible widths, are equally likely to occur. In our experiments, the set W was constructed by finding a lower and upper bound on the induced width for a problem instance. As such, it is more reasonable to assume a Dirichlet prior skewed towards the upper bound. Second, approximating $T_c(w)$ and $T_c(s)$ in the manner described is inherently inaccurate. Complexity analysis of BE suggests the relationship between total state space, induced width and computation time. However, the run-time of BE cannot accurately be predicted as a function of w or s alone. Identifying additional features of an ordering, or combining the features w and s into a single model might improve BE run-time prediction accuracy.

[5]This was computed using the average state space across all mastermind instances.

Induced Width				
Class	**MTQT**	**MI**	**MinMult**	**Fixed**
pedigree	1e-4	1e-5	5 1000	10000
T_{Err}	24.7	-118.7	-154.2	-104.8
(T_{Err})	(910.1)	(622.5)	(596.2)	(381.4)
SDR	2.29	4.11	2.99	2.17
(SDR)	(1.92)	(4.66)	(4.35)	(2.26)
largeFam3	1e-5	1e-5	5 1000	50000
T_{Err}	7572	-4695	-2498	3228
(T_{Err})	(31513)	(7746)	(7546)	(6281)
SDR	3.84	2.4e12	39.5	8.67
(SDR)	(4.03)	(2.6e13)	(250.2)	(67.95)
mastermind	1e-3	1e-5	2 1000	1000
T_{Err}	15616	3493	-3928	-3972
(T_{Err})	(35373)	(14032)	(9303)	(9276)
SDR	1.89	63.3	1.9	1.99
(SDR)	(1.24)	(164.3)	(1.06)	(1.52)

Table 3: Performance of the 4 stopping rules in the induced width experiments. For each problem class we report: 1) T_{Err} averaged across the instances in the problem class; and 2) the average SDR. Standard deviations are shown in parentheses. T_{Err} is reported in second. The parameter settings are listed in the first row of each problem class.

Total State Space				
Class	**MTQT**	**MI**	**MinMult**	**Fixed**
pedigree	1e-2	1e-7	5 100	50000
T_{Err}	556.4	259.9	-363.7	182.3
(T_{Err})	(1046)	(1037)	(1047)	(555.5)
SDR	1.09	1.46	1.41	1.14
(SDR)	(0.16)	(1.38)	(0.55)	(0.20)
largeFam3	1e-10	1e-8	5 5000	50000
T_{Err}	6961	7362	-894.5	-83.3
(T_{Err})	(9249)	(9923)	(11406)	(5291)
SDR	1.31	1.2e9	12.73	2.69
(SDR)	(0.79)	(1.3e10)	(52.31)	(6.97)
mastermind	1e-3	1e-5	2 1000	10000
T_{Err}	-2209	-12621	-16447	-14450
(T_{Err})	(23577)	(27361)	(27545)	(24526)
SDR	1.83	30.98	1.97	4.24
(SDR)	(0.85)	(68.7)	(1.55)	(6.61)

Table 4: Performance on the total state space experiments.

Conclusions

The main contribution of this paper was to formulate the *Stopping Problem*. This allowed us to formalize the trade-off between seeking improved elimination orders and exploiting the best ordering found so far. We then presented three different Stopping Rules for identifying when to discontinue the search for improved orderings and empirically evaluated these rules on a set of Bayes and Markov network instances.

The empirical evaluation showed that the MTQT rule reliably leads to smaller total query response times than ad hoc stopping rules that are used in practice. In many problems, the time savings due to the MTQT stopping rule are substantial. More analysis is needed to understand the graphical model properties favorable for the MTQT rule.

This paper examined stopping rules to minimize the time needed to exactly answer a probabilistic query. In many situations, a rapid approximate response is preferred to an exact answer. This requires responding to a query within a certain amount of time (e.g. 20 seconds). The stopping problem becomes very important in such cases, as the portion of time devoted to seeking good decompositions can affect response accuracy as well. Much work remains to better understand the efficiency-accuracy tradeoff inherent to inference in graphical models. We see the stopping problem as just one part of this important line of research.

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